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CHAPTER 3

Building Mathematics Curricula with Applications and Modelling

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SUMMARY

My perspective is that of a curriculum developer who has for some time been trying to weave applications and modelling into the mathematics curricula of average students. My remarks would be quite different if I were organising courses wholly devoted to applications and modelling, as many have done. I cannot separate applied mathematics from mathematics itself, either in theory or in practice. To me they are both part of the same magnificent edifice.

At international conferences, I am continually surprised by finding that things I thought were the same everywhere are not, but more often things I thought were different in other countries are nearly the same. Still, my perspective was formed in the United States, and I apologise in advance for any parochial views.

The subject of this paper is the building of mathematics curricula with applications and modelling. My remarks and examples are based on my experiences with students in the United States in Grades 7–12, that is, of ages 12–18. For the most part, the students who have been the targets of my work are sitting in classes which they either are required to attend or feel compelled to take for college entrance or success. Most of these students have not made any selection of particular fields of study beyond the notion that they will or will not go to a college or university after completing secondary school. Nearly two-thirds of those who finish secondary school in the United States go to college, almost half of the total age cohort.

I probably do not need to remind you that, although almost all schools in the United States teach nearly the same mathematics curriculum, there is no national curriculum. Furthermore, starting in 9th grade, that is with 14 year olds, the mathematics curriculum is very crowded; only classes with the best students finish their textbooks. This makes change particularly difficult to achieve in grades 9 through 12. Teachers must not only be convinced of the importance of new content, but they have to give up teaching old things in order to have time for the new.

Consequently, when I think of building curriculum in applications and modelling, neither the students nor the teachers I am thinking of are necessarily eager to work with applications and modelling.

The word building has meanings both as a verb and as a noun, and I wish to speak about both these aspects. As a noun, a building is a structure or edifice. If we think of the magnificent edifice of mathematics as a building, with the topics of mathematics being its rooms, then the rooms are interconnected in many ways and this building is very large, perhaps infinite in size. As educators, we have the opportunity to select the doors to the rooms we want to open and the floors of the building upon which we wish students to spend their time. We may lecture on the most interesting features of this building or we may allow students to explore on their own. Some of the rooms of the great edifice of mathematics have doors and windows with wide vistas to the real world; other rooms have small windows to the world and perhaps only one door known to us.

The building of mathematics is very complex, and interrelated with other buildings devoted to the social sciences, the physical sciences, philosophy, business and commerce, and many other domains of human activity. Perhaps a suitable metaphor for the relationships between these buildings is the body. As Mandelbrot (1976) has pointed out, in our bodies there are disjoint networks of arteries, of veins, and of nerves, yet every cell is very close to all of these networks. Similarly, it seems that almost every room of mathematics is close to many many other rooms.

1. THE AIMS OF MODELLING

As a verb, building means constructing. Curriculum theorists generally feel that one should begin the building of a curriculum by agreeing on the aims, goals or objectives of that curriculum. In a plenary session at ICTMA-3, Niss (1989) gave five aims of teaching applications and modelling. Here they are, in a slightly briefer form that he gave them:

(1) to foster creative and problem-solving attitudes, activities, and competences;

(2) to generate a critical potential towards the use and misuse of mathematics in applied contexts;

(3) to provide the opportunity for students to practise applying mathematics that they would need as individuals, citizens, or professionals;

(4) to contribute to a balanced picture of mathematics;

(5) to assist in acquiring and understanding mathematical concepts.

His paper discusses these aims in some detail. I will assume a general agreement with these goals, and will not discuss them further here.

After goals are established, one must decide how to reach them, that is, what to teach, and what experiences to provide for students. Much of the work in the past 20 years in applications and modelling has been devoted to the first step in this process, the collection and exhibition of examples. For instance, at ICME–4 in 1980 my colleague Max Bell was asked to lecture on materials available worldwide for teaching applications of mathematics, and many of the available materials were simply collections of applications (see Bell, 1983). More recently, the proceedings of ICTMA–3 (Blum *et al* 1989) contain numerous articles whose major goal is to point out some beautiful examples of mathematical modelling and applications. The program for ICTMA–4 is similar.

We are in a stage of phenomenal growth of applied mathematics due to the very recent proliferation of personal computers which provide the power to deal with huge amounts of data and heretofore unavailable graphics capability. It is hard to realise that the first personal computers appeared only 12 years ago, in 1977, and the first one–megabyte personal computer is only six years old. Mathematics possesses an accessibility to the general public as never before; that accessibility will cause more and more applications to be found for mathematics. As with pure mathematics, applied mathematics is certain to grow forever. New areas of application are almost certain to be represented at ICTMA–n, for all n.

However, even as this development is in its infancy, the utilisation of applications and mathematical modelling is also beginning its maturity, and as part of that maturation we gain new responsibility. We can no longer be content with the mere display of beautiful applications or of nice examples of modelling. We must organise and sequence the applications so that students gain the general ideas and the power they need to reach the goals summarised by Mogens Niss. If we expect ideas to build in the minds of our students, we must build curricula to match.

Put another way, I believe the time has come for us to consider a longer time frame than just a set of examples or problems, or a single course. If applications and modelling are as important as most of us think they are, then the experiences with these ideas must begin early in a child's education and continue throughout. The questions of selecting, sequencing, and timing these experiences is what I mean by building a curriculum.

2. THE NEED FOR STRUCTURE IN BUILDING A CURRICULUM

I am speaking to the flock, the converted. Not everyone agrees that applications and modelling are so important. These are obstacles to the implementation of applications and modelling. In many countries, including my own, applications of mathematics beyond arithmetic are not part of the standard curriculum. Algebra, geometry, and analysis are taught with few applications. Statistics does not appear. 'Modelling' does not appear even in the index of the books.

There are teachers who think applications are too hard for most students, and so they will teach them only to their best students. And there are teachers who think applications are not really good mathematics, appropriate only for the poorer students as motivation, because the poorer students can't learn good mathematics. The students in the middle, the average students, the majority of students, get fewer experiences with applications than the best or the worst.

I should say that it is from my perspective, perhaps from our perspective, that these students encounter no applications. Many teachers think they are teaching applications through problems like the following, which itself is an example in a widely used 9th grade textbook in the United States.

> A clerk mistakenly reversed the two digits in the price of a marking pen and overcharged the customer 27¢. If the sum of the digits was 15, what was the correct price of the pen?

This problem has at least two characteristics which distinguish it from applications. The first is called *reverse given-find* (Thorndike, 1923). How do you know the sum of the digits is 15 and how do you know the customer was overcharged by 27¢ unless you knew the price in the first place? Reverse given-find is characteristic of many of the word problems which substitute for applications.

Examination of the solution to this problem gives the second characteristic which distinguishes it from an application. The solution begins by letting t = the tens digits and u = the units digit of the sale price. Then it translates the conditions into the equations

$10t + u - (10u+t) = 27$ and $t + u = 15$. After some manoeuvering, this system is solved to find $t = 9$ and $u = 6$. The solution is given in detail – it takes almost a page of the book.

The book does not consider the possibility of using arithmetic. Yet arithmetic provides a more efficient solution. There are only 4 two-digit numbers whose sum of digits is 15; they are 69, 78, 87 and 96. The two of these whose difference is 27 is easy to spot, and so the original price was 69¢ and the overcharge was 96¢. The example *unnecessarily restricts the allowable mathematics,* just the opposite of what would happen in an applied situation.

This problem appears in a 1981 textbook, and I first spoke about it in 1982. I was certain that the publisher would hear about my remarks. You will be interested to know that the problem was changed for the current edition (Dolciani et al, 1986).

> A catalog clerk mistakenly reversed the two digits in the price of a radio fuse and overcharged the customer 36¢. If the sum of the digits was 14, what was the correct price?

It is no wonder that huge numbers of educated adults see no reason to learn algebra, and that there exist newspaper cartoons, in which algebra is portrayed as being only good for one's CV, in which word problems are said to be good not even for the dead, and a famous one (among mathematics aficionados in the United States) in which Hell's library contains many books but all of the same type – story problems.

Most mathematics teachers worldwide received their education when mathematics departments did not have many courses in applied mathematics. Accordingly, to encourage the teaching of applications, it is customary to attempt to acquaint these teachers with large numbers of applications. The result has been virtually imperceptible. Why? Why does knowledge of good applications have so little effect on these teachers? I believe it is because, at the beginning, when they know only a few applications, they do not consider them important. After they learn about greater numbers of applications, there comes a time when they realise that the total number of applications is huge, and that they are relatively ignorant. This overwhelms the average teacher, and every new application merely brings more frustration. The teacher gropes for some way to make applications manageable.

Suppose that just the opposite situation from the present existed, that all the mathematics taught was applied, and none of the abstract properties of operations or of the real numbers were taught. Suppose that we were to desire that some of these properties be learned by students. Suppose we saw some beautiful examples such as the following.

$(a+b)(c+d) = ac + bd + ad + bc.$

$x^n - 1$ is always divisible by $x - 1$.

$1 + nx$ is a good approximation to $(1+x)^n$ if x is small.

Addition is commutative.

If $0 < a < b$ and $0 < c < d$, then $\frac{b}{c} > \frac{a}{d}$

We would be overwhelmed. We need a structure into which these properties fit. (You may already be looking at the last of these and wondering if it is true. You are probably putting it into the structure you already have for these properties.) This is a point I wish to emphasise: *building curricula in applications and modelling requires that we structure the subject.*

3. ORGANISATIONAL STRUCTURES FOR APPLIED MATHEMATICS

We have structures for pure mathematics, structures we associate with famous names: Euclid, Euler, Galois, Boole, and more recently Bourbaki. The structures are logical ones, from postulates through theorems using the vehicles of definitions and proofs. So, although there is a multitude of properties of numbers and operations, people do not feel overwhelmed. We attach new properties to the old structure.

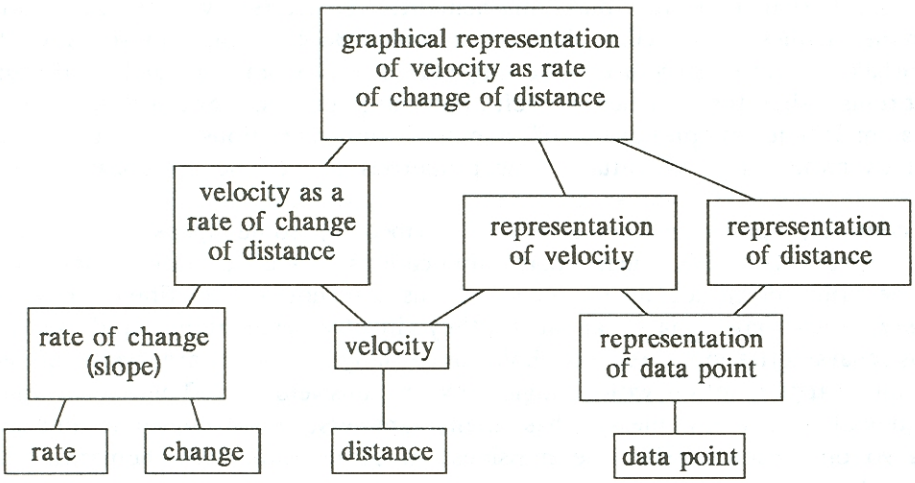
In contract, applied mathematics is described by some speakers as almost the opposite. We hear that applications have exactly what pure mathematics does not have – applications are not well-defined, they are messy, they have many answers, they involve estimation, and so on. This characterisation does not hold universally. There are well-defined, elegant applications with single exact answers. There is pure mathematics that is messy, has many answers, involves approximation, and so on. Emphasising the messiness of some applied mathematics does not help sell this content to traditional teachers.

Teachers need a structure into which they can fit these applications, a structure which is richer than "Here is an application of systems of linear equations". There are many possible structures for applied mathematics. I will describe a few that I have found useful.

Learning Hierarchies

When I first began my work with applications, I wrote a course in which algebra was developed through applications (Usiskin, 1979). The biggest obstacle for my students in learning the applications of algebra was that they did not know the applications of arithmetic. For instance, when discussing population growth and pointing out that a 2% growth rate on a population P yields a population 1.02P, I found that the students had never been given the problem of finding the result of a 2% growth rate on a population of 10,000 or any other size.

A *learning hierarchy* is a network formed in the following way. You examine all the ideas found in an application in order to insure that each component idea is discussed before it is put with the others in the application. For instance, suppose one wishes to consider the graphical representation of velocity as rate of change of distance. You need to know some things about velocity, about rate of change, about distance. Surprisingly many students memorise formulas for the rate of change without having any idea how they are related either to rates or to changes. (In the US we call the rate of change by the word *slope* which disguises the connections; the British term *gradient* is no better.) Their later teachers wrongly assume that the students have learned ideas in earlier courses, ideas which they have never studied. Here is a possible learning hierarchy for this idea.



This is a natural hierarchy, but what makes it difficult to implement is that the components occur over many years of a student's schooling. For instance, the idea of change could be taught with subtraction as early as second grade. This may explain why such hierarchies are ignored by teachers. They do not want to go back to first principles, viewing that as a waste of time. Yet, when students cannot do an application, it is often because there is something down the hierarchy which they do not understand or which they have not been taught, and for that student it is pointless to go on.

Use Meanings

The bottom row of the hierarchy contains some ideas which are intimately related to particular mathematical concepts. Rate, as in kilometers per hour, or students per class, or population per square kilometer, is a fundamental use of division. Change is an application of subtraction. Distance can be considered as another application of subtraction; at higher levels distance is an application of absolute value. Data point is a fundamental use of ordered n–tuples. One of the foundations of a curriculum in applications and mathematical modelling has to be the giving attention to the basic uses of the commonly taught or most important mathematical concepts. Active attention to these uses is in the curriculum being developed by the University of Chicago School Mathematics Project (1989).

For example, the idea of point has at least five use meanings, four of which are found in the UCSMP (1990) materials: dot, the meaning held by children and used on computer and television screens; idealised location, the usual meaning in geometry; ordered pair or triple or n–tuple, the usual meaning in numerical applications and in algebra, and the model for data point; node, the meaning found in networks and graphs; and centre of gravity, the meaning needed for work with physical forces. The mathematics of point is different in these various guises, and too often one of these meanings so predominates that the others are considered distasteful. For instance, students may be instructed that dots are not mathematical points. However, for applied mathematics, one must consider pixels as points. The student who thinks points must be locations is thwarted in understanding n–space when $n > 3$. Yet that is a natural occurrence with data points.

There are use meanings of more advanced concepts. Linear functions seem to have two basic use meanings: linear combination and constant increase/decrease. That is, they arise from those kinds of situations in the real world. They also arise from the uses of lines as shortest distances, as outlines or intersections of geometric figures, and as approximations to curves. The trigonometric functions also have at least three basic uses: one has to do with ratios of lengths in triangles; one has to do with circular motion and periodicity; and one has to do with acceleration and differential equations.

Processes also have use meanings. For instance, the process of estimation is used for many reasons: clarity, for ease of understanding; facility, for ease of use; consistency, to agree with other precisions; economy, to save time; safety, as when we estimate the maximum weight an airplane can hold; and finally the many situations in which estimates are forced.

Freudenthal (1983) has stressed the need for multiple characterisations of mathematical concepts if one is to understand their relationships to the real world. I agree completely. When Max Bell and I engaged in an in-depth study of the uses of arithmetic (1983), we developed six meanings for number; count, measure, location, ratio comparison, code, or derived constant. The location category includes addresses, the ordinals, temperatures, and any other numbers on scales. We found that scales like those used for measuring earthquakes or star magnitudes simply did not appear in the curriculum at any level, and when temperature was taught, it was often treated as if it were a measure like volume. We would see problems like the following in books: "If the temperature is 2° and it triples, what will be the new temperature?" Confusions in the use of mathematical objects cannot help if one wishes students to learn applied mathematics.

It is obvious that learning hierarchies and use meanings are somewhat related. The use meanings tell you what to put at the bottom of the learning hierarchy for applications; they tell you what should come first in the curriculum.

Analogies with Pure Mathematics

Building a curriculum in applied mathematics is somewhat of an unsolved problem. Following the advice of George Polya, one way to tackle this problem is to seek analogous problems which we have solved. This suggests that one way to build a curriculum in applied mathematics is to take advantage of its analogies with pure mathematics.

The use meanings discussed earlier are analogous to postulates or definitions in a mathematical system. Combining rate and change to get rate of change is akin in combining postulates and definitions to get a theorem.

I consider the process of modelling as being quite analogous – at least in a pedagogical sense – to proof. Each is at the highest realm of activity. Each is at the highest level of cognitive activity. Do we want students to learn proof, because that is what mathematicians do? If so, then we should want students to learn modelling, because that is what applied mathematicians do. If we do not want students to learn proof, then perhaps we should not want them to learn to model. My own view is that we do want students to learn some aspects of both proof and modelling, to have experiences with both.

We have been teaching (or trying to teach) students proof for quite a bit longer than we have been teaching modelling. What can we learn from that experience? Here are some thoughts.

It is almost impossible to understand proofs with concepts just introduced. For instance, the student who is just introduced to the concept of group finds it very difficult to write proofs about groups immediately. Similarly, modelling in contexts not understood by the students is likely to be a waste of time.

The ability to do proofs in one domain does not necessarily extend to the ability to do proofs in another domain. The same is true for modelling. That suggests that students should learn a variety of kinds of modelling.

Proof learning required the language of if-then statements and justifying conclusions. Modelling requires the same logic – if this model is assumed, then we can deduce ... Thus we should be able to use proof to teach modelling and use modelling to teach proof.

Communication of proofs and modelling both require that students write using both mathematical and non-mathematical terminology. Independent of the mathematical concepts, the writing task itself poses difficulties for many students.

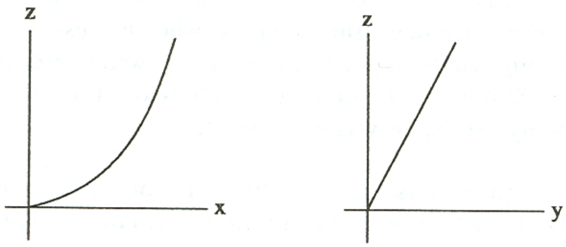
Proof competence comes quite slowly. We should not expect modelling competence to come any more quickly.

Some proofs, like the infinitude of primes or the irrationality of $\sqrt{2}$, are classic. Similarly, some models should be treated with the same reverence, and taught to all. One that would surely qualify is Kepler's modelling of Tycho Brahe's data with ellipses. Another is the simulation of coin-tossing.

Dimensions of Understanding

A fourth structure also relates applied mathematics to other mathematics, through the following question. What is meant by *understanding* a particular concept?

Consider the concept of joint variation, as in the formula $z = kx^2y$. To many people, you understand joint variation if you can calculate the value of k given values of x, y, and z. Understanding is doing; it involves skill and algorithm. A second sort of understanding consists of the mathematical theory: why is z quadrupled when x is doubled and y is kept constant? Understanding is knowing why; it is the mathematical underpinnings dimension. A representational sort of understanding is favored by many psychologists.



To them, understanding is being able to present or to find metaphors. Many in applied mathematics view a fourth type of understanding as the final goal: Do you know when and how apply the idea? In the case of $z = kx^2y$, one would at least want to consider the formula $V = \pi r^2h$ for the volume of a cylinder, and discuss all of the other aspects in terms of their real world implications.

These four major dimensions of understanding each have simple aspects, and each have more complicated aspects. The skill–algorithm dimension ranges from memorised basic facts through the carrying out of procedures to the invention of algorithms. The mathematical properties underpinning dimension ranges from being able to name properties to justifications with them and, at the highest level, the discovery of proofs. The use–application dimension ranges from straightforward one-step uses through the applications of mathematical principles to modelling. The representation–metaphor dimension ranges from concrete materials to the use of representations to the creation of new metaphors or representations.

One can view the new math as an attempt to assert the importance of mathematical underpinnings. 'Back to basics' was an attempt to reassert skill. The popularity of Piaget and concrete materials has been a recognition of the importance of representations in understanding. The existence of the ICTMAs and moves towards applications and modelling are the assertion of the importance of the use dimension.

I believe students are best served by the view that all these dimensions contribute to the 'real' understanding of mathematical concepts. Furthermore, viewing understanding as having many dimensions helps to operationalise Niss's last two reasons for teaching applications and modelling, namely it contributes to a balanced picture of mathematics and assists the student in acquiring an understanding of mathematical concepts.

Closeness of Fit of Models

The process of modelling can be described as an attempt to find mathematical concepts which are isomorphic to situations in the real world. The utility of the model is a function of the degree of isomorphism, the closeness of the fit of the model. My experience suggests that one should first consider models which are isomorphic and move gradually to those which are not. For instance, in considering situations modelled by quadratic functions, one might move first from an *exact* model, for example the number of games required for n teams to play each other; to an *almost-exact theory-based* model, for example the path of a thrown, kicked, or batted ball; to what could be called an *impressionistic* model, for example the use of a parabola to model a frequency distribution.

We do not seem to have a language to describe the closeness of fit. We call all of them models, not distinguishing between economic models, which are surely impressionistic given their standard of performance, and the almost-exact models found in the mathematics of celestial navigation. I hope someone can come up with better words than the ones I have used (exact, almost exact theory based, impressionistic). It would help the curriculum if we had a better language for describing the fit of models. (I was told recently that the word *phenomenological* is often used where I have used impressionistic.)

4. BUILDING WITH TECHNOLOGY

No one can consider building a curriculum without considering the role of technology; technology has driven the revolution in applied mathematics. The usual things to say in this regard are: the cost of hardware continues to decrease, the software is getting better and better, there is good stuff out there and bad stuff out there. You will see some wonderful things demonstrated at this meeting. There – I've said them.

I would like to discuss briefly some issues less often discussed: the conflict with traditional ways, logistics, and equity.

The Conflict with Traditional Ways

The conflict with tradition is epitomised in answers given to the following question. What should a student be able to do without the aid of technology? It is a fundamental question, complicated because technology has greatly expanded what students can do. The responses occasionally defy common sense.

There are those who believe that students should be required to have exactly the same paper and pencil arithmetic skills they needed some time ago despite the existence of calculators. There are those who believe that students should still be expected to get integrals by hand even if they have access to computer software which deals with symbols. The justification is that real understanding does not occur without labour.

Yet we know that many students have the skills without any of the other kinds of understandings. One reason for imposing a structure on understanding is to explain the incorrectness of the view that one type of understanding is always necessary for another.

On the other hand, there are those who believe that technology will allow paper-and-pencil skills of students to be maintained even if less time is spent on them. This runs contrary to my own common sense; virtually all educational research supports strong effects for time-on-task.

A compromise position is to require students to do the pure problems by hand but allow technology on the applied problems. This too is illogical. If the technology is there, why not use it all the time?

Curiously, the weaker technological tool – the simple calculator – is seen as more threatening than the computer. But I believe it is because the user-friendly symbol-pushing calculator is not yet here. Once skills from algebra through differential equations become automatic, we will see teachers banning their use. Many teachers are quite willing to have others use technology but unwilling to allow their own students to use it.

Logistics

Logistics, by which I mean the scheduling of student and machine, may contribute to the unwillingness of many teachers to utilise computer technology. In most schools, it is impossible to have all students at computers at all times. The computer technology facilities thus have to be scheduled. Though it is more difficult to schedule sporadic use of a computer than to schedule continual use, the scheduling of continual use may not be warranted by the range of the mathematics content in the course.

The logistics problem is particularly critical on timed examinations. There is a natural tendency to give harder tasks to students when the students have access to calculators or computers. Otherwise, why use this equipment? However, working the technology takes time, and if the technology does any of the work, it takes quite a bit of time for the students to explain what was done. The management of evaluation with technology is in its infancy.

The logistics problem also appears with out-of-class assignments. If a student is living in a place with no access to computers, how can an assignment be given that requires computers?

Equity

Pencils and paper are inexpensive; if a student does not have them, they can usually be obtained. But a computer is much more expensive, and a student in today's mathematical world who has a computer equipped with sophisticated software at home is at an enormous advantage over one who doesn't. Thus the greater accessibility of mathematics has brought with it the troubling thought that, until all in a society have access to technology, the poor will be even more disadvantaged.

Hardware
When both a calculator and computer can do a job, even though the calculator does not do it as well, the wisdom is beginning to be amassed to use the calculator. For instance, Bert Waits and Franklin Demana at Ohio State University now encourage use of a calculator function grapher rather than their own very fine computer function graphing software for secondary school classes, because every student can take home a calculator. This also serves to solve the equity issue.

Some lap-tops are not full personal computers, have limited memory, and are meant to be used in connection with a larger computer. These are the modern-day slates and may become the student calculator of the future.

5. CLOSING REMARKS

I would like to put the computer revolution in perspective. It is truly a revolution, unlike anything in mathematics since the development of the written algorithms for arithmetic in the 15th century. At that time, algorithms for addition and subtraction of whole numbers had been known for some time, but algorithms for multiplication and division like those we now teach in elementary school had just been invented. Before this time, it is reported (Schwetz, 1987) that some arithmetic students were required to learn the multiplication tables from 1×1 to 99×99 because they had no other option for quick multiplication. Their only other option was to use lots of addition.

It was in Italy that they first taught these new algorithms for multiplication and division, and they were taught to students of secondary school and university age. This was the applied mathematics of the time, driven by the needs of the merchants and traders of the Italian city-states.

If these algorithms were the new mathematics, the new technology was printing. Before printing, universal literacy was non-existent. There weren't enough books to teach the masses how to read. Reading was considered a difficult chore, capable of being mastered only by an aristocracy or by the specially ordained. We all know that stories were passed down orally from generation to generation. I've wondered why they weren't written down - they seem to have been passed down even in cultures that had some writing. I'm convinced that people felt that something would be lost in the writing – the meter, the pauses, the music possibly. Those who could not read and write did not trust this new technology of reading and writing.

But printing changed all that. Now everyone could have books, even in their homes. And so the fear and mistrust of reading and writing waned, and people could aim for full literacy.

I believe the same will happen for mathematics. Calculators already are in almost every home in the industrialised countries, and computers in increasing percentages of them. These are the 20th century equivalents of books, and they are taking the fear out of numbers, out of graphs, and out of symbolism. The public is slowly learning that mathematics is simply a particular kind of language, the language of the logic and structure of phenomena in the universe – and that when mathematics is around, mathematics can be learned by all who learn to read as early as they learn to read.

Thus we can expect greater and greater amounts of applied mathematics to be a part of the experiences of all people whether or not they learn it in school. We all want students to be able to recognise situations that call for mathematical techniques, to encourage the wise use of the mathematics, and to discourage improper use. The fundamental principles must begin early, for a building is no stronger than its foundation. I hope that my remarks have given a few ideas on how that building might be arranged.

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